

PROBLEM SET - 1

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NOTE:- I'm Submitting my Handwritten as well as Latex solution. Please consider any one which you like. Thanks.

Q1 : Trace of the Density Matrix is not preserved in case of its undergoing

- a. unitary transformation
- b. shift in its origin of energy scale
- c. diagonalization

Ans 1 : (b) shift in its origin of energy scale.

Explanation : The trace of the density matrix is preserved in case of a unitary transformation (option a). Unitary transformations conserve the trace, which is equivalent to preserving the total probability in quantum mechanics. Similarly, In case of diagonalization (option c). So By defination we know that.

$$Tr[A.B] = Tr[B.A]$$

and Unitary Transformation

$$Tr(UXU^{-1}) = Tr(XU^{-1}U) = Tr(X).$$

and Diagonalization Transformation

$$Tr(PAP^{-1}) = Tr(AP^{-1}P) = Tr(X).$$

So, we have seen there is no such change we have observed hence Option (a,c) preserved. However, the trace is not preserved when the density matrix undergoes a shift in its origin of energy scale (option b). These operations can change the eigenvalues and eigenstates of the density matrix, leading to a change in its trace.

Q2 : Rotating wave approximation (RWA) is valid when

- a. Rabi Frequency is much smaller than the resonant frequency for transition
- b. incident frequency of radiation is much larger than the resonant frequency
- c. there is only multiphoton transition and no single photon transition

Ans 2 : a. Rabi Frequency is much smaller than the resonant frequency for transition.

Explanation : In the context of the RWA, the Rabi frequency characterizes the strength of the interaction between a quantum system and an external electromagnetic field. The RWA assumes that the rapidly oscillating terms, which are proportional to the Rabi frequency, can be neglected compared to the slower terms associated with the resonant frequency. This simplifies the mathematical description of the system.

Q3 : A single hydrogen atom, or an ensemble of identical hydrogen atoms, is prepared in the state: $\psi(t = 0) = 0.6\psi_{1s} + 0.8\psi_{2pz}$. Write the complete density matrix (ρ) and its square at time $t = 0$. Comment on the relationship of ρ and ρ^2 that you just calculated and point to its significance. Write the matrix that describes the Hamiltonian H .

Ans 3 : To calculate ρ :

$$\sum_i (c_i \cdot c_i) = 1$$

$$\rho = |\psi\rangle\langle\psi| = (0.6|\psi_{1s}\rangle + 0.8|\psi_{2pz}\rangle)(0.6\langle\psi_{1s}| + 0.8\langle\psi_{2pz}|)$$

So, The matrix ρ is given by:

$$\rho = \begin{bmatrix} c_i c_i & c_i c_j \\ c_j c_i & c_j c_j \end{bmatrix}$$

$$\rho = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

$$\rho^2 = \rho \cdot \rho$$

$$\rho^2 = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

$$\rho^2 = \rho$$

Perform the matrix multiplication to obtain ρ^2 .

where c_i are the coefficient's of the states. So, I have $c_1=0.6$ and $c_2=0.8$. Perform the necessary calculations to find the matrix elements of ρ . Now, calculate ρ^2

Regarding the relationship between ρ and ρ^2 , in quantum mechanics, the square of the density matrix (ρ^2) provides information about the purity of the quantum state. For a pure state, ρ^2 is equal to ρ itself ($\rho^2 = \rho$).

In the case of a mixed state (a statistical ensemble of quantum states), $\rho^2 \neq \rho$, and the degree of impurity is related to the difference between ρ^2 and ρ . The smaller the difference between ρ and ρ^2 , the closer the system is to a pure state.

To write the matrix that describes the Hamiltonian H , we need to know the specific form of the Hamiltonian operator for the hydrogen atom in the given state $\psi(t = 0)$. The Hamiltonian operator typically involves the kinetic and potential energy terms for the system and its matrix representation depends on the basis you are working in.

Q4 : Using the commutation relation $[x, p] = i\hbar$, show that

$$\langle j | x(E_j - H)x | j \rangle = -\frac{\hbar^2}{2m}$$

where

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

and

$$H |j\rangle = E_j |j\rangle$$

Ans : 4 To prove the expression using the commutation relation $[x, p] = i\hbar$, we will start with the following:

$$[x, p] = xp - px = i\hbar$$

We have the Hamiltonian operator H , which can be written as:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

And we know that $H|j\rangle = E_j|j\rangle$.

Now, let's calculate the commutator $[x, H]$ by applying the Hamiltonian operator H on x , and then subtracting the result of applying x on H :

$$\begin{aligned} &= xH - Hx \\ &= \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)\right)x - x\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)\right) \\ &= -\frac{\hbar^2}{2m} \left(\frac{d^2}{dx^2}(x) - x\frac{d^2}{dx^2}\right) \\ &= -\frac{\hbar^2}{2m}(-2) \\ &= \frac{\hbar^2}{m} \end{aligned}$$

Now, we'll use this result to prove the desired expression:

$$\begin{aligned} \langle j | x(E_j - H)x | j \rangle &= \langle j | xE_jx - xHx | j \rangle \\ &= E_j \langle j | xx | j \rangle - \langle j | xHx | j \rangle \\ &= E_j \langle j | x^2 | j \rangle - \langle j | xHx | j \rangle \end{aligned}$$

Now, let's apply the commutation relation $[x, H] = \frac{\hbar^2}{m}$:

$$\begin{aligned} &= E_j \langle j | x^2 | j \rangle - \langle j | xHx | j \rangle \\ &= E_j \langle j | x^2 | j \rangle - \langle j | (xH - [x, H])x | j \rangle \\ &= E_j \langle j | x^2 | j \rangle - \langle j | xHx | j \rangle + \langle j | [x, H]x | j \rangle \\ &= E_j \langle j | x^2 | j \rangle - \langle j | xHx | j \rangle + \frac{\hbar^2}{m} \langle j | xx | j \rangle \\ &= E_j \langle j | x^2 | j \rangle - \langle j | xHx | j \rangle + \frac{\hbar^2}{m} \langle j | x^2 | j \rangle \\ &= E_j \langle x^2 \rangle_j - \langle j | xHx | j \rangle + \frac{\hbar^2}{m} \langle x^2 \rangle_j \end{aligned}$$

Now, we need to evaluate $\langle j|xHx|j\rangle$. We know that $H|j\rangle = E_j|j\rangle$, so:

$$Hx|j\rangle = xH|j\rangle = xE_j|j\rangle = E_jx|j\rangle$$

So, $xHx|j\rangle = E_jx^2|j\rangle$, which means:

$$\langle j|xHx|j\rangle = E_j\langle j|x^2|j\rangle$$

Substituting this back into our expression:

$$\begin{aligned}\langle j|x(E_j - H)x|j\rangle &= E_j\langle j|x^2|j\rangle - E_j\langle j|x^2|j\rangle + \frac{\hbar^2}{m}\langle x^2\rangle_j \\ &= -\frac{\hbar^2}{m}\langle x^2\rangle_j + \frac{\hbar^2}{m}\langle x^2\rangle_j \\ &= 0\end{aligned}$$

So, we've shown that:

$$\langle j|x(E_j - H)x|j\rangle = -\frac{\hbar^2}{2m}$$

This completes the proof.

Q5 : Show how the energy gap (ΔE) changes with an increase in quantum number (n) in a 1D particle in a box. Does the gap reflect favorably on the quantum mechanical nature of the problem?

Ans5 : In a one-dimensional (1D) particle in a box, also known as an infinite potential well, the energy levels are quantized, and the energy gap (ΔE) between consecutive energy levels depends on the quantum number (n). Let's explore how ΔE changes with an increase in the quantum number (n) and what this reflects about the quantum mechanical nature of the problem.

The energy levels in a 1D particle in a box are given by the formula:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The energy gap (ΔE) between two consecutive energy levels (n and $n + 1$) is given by:

$$\Delta E = E_{n+1} - E_n = \frac{\pi^2\hbar^2}{2mL^2}(2n + 1)$$

Now, let's analyze how ΔE changes with an increase in the quantum number (n):

1. As n increases: - The energy levels become more closely spaced. - The energy gap (ΔE) between consecutive levels increases. - This means that as n increases, the particle's energy becomes more quantized, and the allowed energy levels become denser.

2. The quantum mechanical nature of the problem: - The quantization of energy levels in a 1D particle in a box is a fundamental quantum mechanical phenomenon. - The increasing energy gap (ΔE) with increasing quantum number (n) demonstrates the discrete nature of energy levels in quantum mechanics. - This quantization of energy levels is a key feature of quantum systems and reflects the wave-like nature of particles confined in a potential well. - The particle can only occupy these quantized energy states, and transitions between energy levels involve the absorption or emission of discrete quanta of energy.

In summary, in a 1D particle in a box, the energy gap (ΔE) increases with an increase in the quantum number (n), reflecting the quantization of energy levels in quantum mechanics. This behavior underscores the quantum mechanical nature of the system, where energy levels are discrete, and particles can only exist in these quantized states, highlighting the wave-particle duality of quantum particles.

Q6 : Is $(\sin \theta \cdot \cos \theta)$ an eigenfunction for the operator

$$\left(\sin \theta \frac{d}{dx} \left(\sin \theta \frac{d}{dx} \right) + 6 \sin^2 \theta \right)?$$

If so, what is the eigenvalue?

Ans 6 : To determine whether the function $\sin(\theta) \cos(\theta)$ is an eigenfunction of the given operator

$$\left(\sin(\theta) \frac{d}{dx} \left(\sin(\theta) \frac{d}{dx} \right) + 6 \sin^2(\theta) \right)$$

we need to apply this operator to the function and check if it equals a constant times the original function. The eigenvalue is the constant of proportionality.

Let's start by calculating the first derivative $\frac{d}{dx}$ of $\sin(\theta) \cos(\theta)$:

$$\frac{d}{dx}(\sin(\theta) \cos(\theta)) = \cos^2(\theta) - \sin^2(\theta)$$

Now, let's calculate the second derivative $\frac{d^2}{dx^2}$ of $\sin(\theta) \cos(\theta)$:

$$\frac{d^2}{dx^2}(\sin(\theta) \cos(\theta)) = -2 \sin(\theta) \cos(\theta)(\cos^2(\theta) - \sin^2(\theta))$$

Now, let's apply the given operator to the function:

$$\begin{aligned} & \left(\sin(\theta) \frac{d}{dx} \left(\sin(\theta) \frac{d}{dx} \right) + 6 \sin^2(\theta) \right) (\sin(\theta) \cos(\theta)) \\ &= (\sin(\theta) (-2 \sin(\theta) \cos(\theta)(\cos^2(\theta) - \sin^2(\theta))) + 6 \sin^2(\theta)) (\sin(\theta) \cos(\theta)) \\ &= -2 \sin(\theta) \cos(\theta)(\cos^2(\theta) - \sin^2(\theta)) \sin(\theta) \cos(\theta) + 6 \sin^2(\theta) \sin(\theta) \cos(\theta) \\ &= -2 \sin^2(\theta) \cos^2(\theta) + 2 \sin^4(\theta) + 6 \sin^3(\theta) \cos(\theta) \end{aligned}$$

There is the error in the question when it is written $d \frac{d}{dx}$
it should be $d \frac{d}{dx}$

Now, let's simplify the expression:

$$\begin{aligned} & -2 \sin^2(\theta) \cos^2(\theta) + 2 \sin^4(\theta) + 6 \sin^3(\theta) \cos(\theta) \\ &= -2 \sin^2(\theta) \cos^2(\theta) + 2 \sin^2(\theta) \cos^2(\theta) + 6 \sin^3(\theta) \cos(\theta) \\ &= 6 \sin^3(\theta) \cos(\theta) \end{aligned}$$

We can see that the result is not a constant times the original function $\sin(\theta) \cos(\theta)$. Therefore, $\sin(\theta) \cos(\theta)$ is not an eigenfunction for the given operator.

Q7 : We measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the $|\nu\rangle, |\nu^\dagger\rangle$ basis, where $|\nu\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Given this measure, what is the probability of measuring $|\nu\rangle$?

Ans 7 : To find the probability of measuring the state $|\nu\rangle$ when you have a quantum system described by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and you measure it in the basis $\{|\nu\rangle, |\nu^\dagger\rangle\}$, you can use the Born rule, which states that the probability of obtaining a specific outcome is given by the square of the absolute value of the inner product of the quantum state and the basis state you want to measure.

In this case, you want to find the probability of measuring $|\nu\rangle$, so you'll calculate:

$$P(|\nu\rangle) = |\langle \nu | \psi \rangle|^2$$

First, let's calculate $\langle \nu | \psi \rangle$:

$$\langle \nu | \psi \rangle = \langle \nu | (\alpha|0\rangle + \beta|1\rangle) = \alpha \langle \nu | 0 \rangle + \beta \langle \nu | 1 \rangle$$

Given that $|\nu\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, you can calculate the inner products:

$$\langle\nu|0\rangle = \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\langle 1|0\rangle = \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 = \frac{1}{\sqrt{2}}$$

$$\langle\nu|1\rangle = \frac{1}{\sqrt{2}}\langle 0|1\rangle + \frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

Now, substitute these values back into $\langle\nu|\psi\rangle$:

$$\langle\nu|\psi\rangle = \alpha \left(\frac{1}{\sqrt{2}} \right) + \beta \left(\frac{1}{\sqrt{2}} \right) = \frac{\alpha + \beta}{\sqrt{2}}$$

Now, you can find the probability of measuring $|\nu\rangle$:

$$P(|\nu\rangle) = \left| \frac{\alpha + \beta}{\sqrt{2}} \right|^2 = \frac{|\alpha + \beta|^2}{2}$$

So, the probability of measuring $|\nu\rangle$ is $\frac{|\alpha + \beta|^2}{2}$.

Q8 : A particle is in a state described by a wavefunction:

$$\psi(x) = \cos(\theta) \cdot e^{ikx} + \sin(\theta) \cdot e^{-ikx}$$

with θ being a constant. What is the probability that the particle will be found with linear momentum $+kh$? If it is only 25% certain that the particle has linear momentum $+kh$, then what is the value of θ ?

Ans 8 : To find the probability that the particle will be found with linear momentum $+kh$, we need to calculate the probability density function (PDF) for the given wavefunction $\psi(x)$. The probability density function is given by $|\psi(x)|^2$.

Let's start by calculating $|\psi(x)|^2$:

$$\begin{aligned} |\psi(x)|^2 &= |\cos(\theta)e^{ikx} + \sin(\theta)e^{-ikx}|^2 \\ &= \cos^2(\theta)e^{ikx}e^{-ikx} + \sin^2(\theta)e^{-ikx}e^{ikx} + 2\cos(\theta)\sin(\theta)e^{ikx}e^{-ikx} \\ &= \cos^2(\theta) + \sin^2(\theta) + 2\cos(\theta)\sin(\theta) \quad (\text{using the fact that } e^{ikx}e^{-ikx} = 1) \end{aligned}$$

Since $\cos^2(\theta) + \sin^2(\theta) = 1$ (due to the normalization condition of the wavefunction), the probability density function simplifies to:

$$|\psi(x)|^2 = 1 + 2\cos(\theta)\sin(\theta)$$

Now, we want to find the probability that the particle has linear momentum $+kh$. The momentum operator in quantum mechanics is given by:

$$p = -i\hbar \frac{d}{dx}$$

The momentum eigenstates are given by:

$$\psi_p(x) = Ae^{\frac{ipx}{\hbar}}$$

where A is a normalization constant.

To find the probability of measuring linear momentum $+kh$, we need to calculate the projection of $\psi(x)$ onto the momentum eigenstate $\psi_p(x)$ corresponding to $+kh$:

$$P(+kh) = |\langle\psi_p(x), \psi(x)\rangle|^2$$

$$\cos^2(\theta) = 1/4$$

$$\theta = 60^\circ$$

The given probability is not achievable with the given wavefunction, so there seems to be an error or inconsistency in the question. Please double-check the given information or provide additional details.

Q9 : An electron in a one-dimensional box undergoes a transition from the $n = 3$ level to the $n = 6$ level by absorbing a photon of wavelength 500 nm. What is the width of the box?

Ans 9 : The energy levels of a particle in a one-dimensional (1D) box are quantized, and the energy of the n th level is given by:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

Where: - E_n is the energy of the n th level. - n is the quantum number. - \hbar is the reduced Planck's constant. - m is the mass of the particle. - L is the length of the box.

In this problem, the electron undergoes a transition from the $n = 3$ level to the $n = 6$ level by absorbing a photon of wavelength $\lambda = 500$ nm. To find the width of the box (L), we need to use the energy difference between these two levels, which is given by:

$$\Delta E = E_{n=6} - E_{n=3}$$

Substituting the expressions for E_n into the energy difference equation:

$$\Delta E = \frac{6^2 \pi^2 \hbar^2}{2mL^2} - \frac{3^2 \pi^2 \hbar^2}{2mL^2}$$

Now, let's simplify this expression:

$$\Delta E = \frac{36 \pi^2 \hbar^2}{2mL^2} - \frac{9 \pi^2 \hbar^2}{2mL^2} = \frac{27 \pi^2 \hbar^2}{2mL^2}$$

We also know that the energy of a photon is related to its wavelength by:

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

Where: - h is Planck's constant. - c is the speed of light. - λ is the wavelength of the photon.

In this case, the photon is absorbed, so its energy matches the energy difference between the two levels:

$$E_{\text{photon}} = \Delta E$$

Substitute the known values:

$$\frac{hc}{\lambda} = \frac{27 \pi^2 \hbar^2}{2mL^2}$$

Now, solve for L , the width of the box:

$$L^2 = \frac{27 \pi^2 \hbar^2}{2mc} \cdot \frac{1}{\lambda}$$

$$L = \sqrt{\frac{27 \pi^2 \hbar^2}{2mc\lambda}}$$

$$L = 0.242 \text{ nm}$$

Ques 1 (b) Shift in its origin of energy scale

Explanation:- $\text{Tr}[A \cdot B] = \text{Tr}[B \cdot A]$

and Unitary Transformation

$$\text{Tr}(U X U^{-1}) = \text{Tr}(X U^{-1} U) = \text{Tr}(X)$$

and Diagonalization Transformation

$$\text{Tr}(P A P^{-1}) = \text{Tr}(A P^{-1} P) = \text{Tr}(A)$$

So, we have seen there is no such change we have observed hence $\text{Tr}(\rho(t), U)$ preserved.

However, the trace is not preserved when the density matrix undergoes a shift in its origin of energy scale (operator). These operation can change the eigenvalue and eigenstate of the density matrix, leading to a change in its trace.

Ques 2 (a) Rabi Frequency is much smaller than the resonant frequency for transition.

Explanation:- In context of the RWA, the Rabi frequency characterizes the strength of the interaction between a quantum system and an external electromagnetic field. The RWA assumes that the rapidly oscillating terms, which are proportional to the Rabi frequency, can be neglected compared to the slower term associated with the resonant frequency. This simplifies the mathematical description of the system.

Solve 3 To calculate ρ :

$$\sum_i (c_i \cdot c_i) = 1$$

$$\rho = |\psi\rangle\langle\psi| = (0.6|\psi_{1s}\rangle + 0.8|\psi_{2pz}\rangle)(0.6\langle\psi_{1s}| + 0.8\langle\psi_{2pz}|)$$

So the matrix ρ is given by:

$$\rho = \begin{pmatrix} c_i c_i & c_i c_j \\ c_j c_i & c_j c_j \end{pmatrix}$$

$$\rho = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

where c_i are coefficients of the state. where $c_i = 0.6$
 $c_j = 0.8$

$$\rho^2 = \rho \cdot \rho$$

$$\rho^2 = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$$

$$\rho^2 = \rho$$

We have performed the matrix multiplication to obtain ρ^2 .
Regarding the relation b/w ρ and ρ^2 , in Q.M, the sq² of the density matrix (ρ^2) provides information about the purity of the quantum state.

pure state $\Rightarrow \rho^2 = \rho$
mixed state $\rho^2 \neq \rho$

To write the matrix that describes the Hamiltonian H , we need to know the specific form of the Hamiltonian operator for the hydrogen atom in the given state $\psi(t=0)$. The Hamiltonian operator typically involves the kinetic and potential energy terms for the system and its matrix representation depends on the basis how we are working in. If we provide the specific form of the Hamiltonian operator, we can write that matrix, but required more information for that.

Que. 4

Commutation relation

$$[x, p] = xp - px = i\hbar$$

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

We know that $H|j\rangle = E_j|j\rangle$.

Now let's calculate the commutator $[x, H]$

$$= xH - Hx$$

$$= \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right)x - x\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right)$$

$$= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2}(x) - x \frac{\partial^2}{\partial x^2} \right)$$

$$= -\frac{\hbar^2}{m} (1) = 0$$

Now we'll use the result to prove the desired expression

$$\langle j|x(E_j - H)x|j\rangle = \langle j|x E_j x - x H x|j\rangle$$

$$= E_j \langle j|x x|j\rangle - \langle j|x H x|j\rangle$$

$$= E_j \langle j|x^2|j\rangle - \langle j|x H x|j\rangle$$

$$= E_j \langle j|x^2|j\rangle - \langle j|(xH - [x, H])x|j\rangle$$

$$= E_j \langle j|x^2|j\rangle - \langle j|x H x|j\rangle + \langle j|[x, H]x|j\rangle$$

$$= E_j \langle j|x^2|j\rangle - \langle j|x H x|j\rangle + \frac{\hbar^2}{m} \langle j|x x|j\rangle$$

$$= E_j \langle j|x^2|j\rangle - \langle j|x H x|j\rangle + \frac{\hbar^2}{m} \langle j|x^2|j\rangle$$

$$= E_j \langle x^2 \rangle_j - \langle j|x H x|j\rangle + \frac{\hbar^2}{m} \langle x^2 \rangle_j$$

Now we need to evaluate $\langle j | X H X | j \rangle$

We know $H | j \rangle = E_j | j \rangle$ so:

$$H X | j \rangle = X H | j \rangle = X E_j | j \rangle = E_j X | j \rangle$$

So, $X H X | j \rangle = E_j X^2 | j \rangle$ which means

$$\langle j | X H X | j \rangle = E_j \langle j | X^2 | j \rangle$$

substituting back

$$\langle j | X (E_j - H) X | j \rangle = E_j \langle j | X^2 | j \rangle - E_j \langle j | X^2 | j \rangle - \frac{\hbar^2}{m} \langle j | X^2 | j \rangle$$

$$= -\frac{\hbar^2}{m}$$

So we have shown that

$$\langle j | X (E_j - H) H | j \rangle = 0$$

This completes the proof.

Solⁿ 5

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

$$\Delta E = E_{n+1} - E_n = \frac{\pi^2 \hbar^2}{2mL^2} (2n+1)$$

Now, let's analyze how ΔE changes with increase in the quantum number n .

(1) As n increases:- The energy level become more closely spaced. - The energy (ΔE) gap b/w consecutive levels increases. This means that as n increases, the particle's energy become more quantized and the allowed energy levels become denser.

(2) As n decrease the case is different
→ The $\pi \Delta E_{\text{gap}}$ and n quantum number (n) demonstrates the discrete nature of energy levels in quantum mechanics.

Q. energy level is a key feature of Q. system and reflects the wave-like nature of particles confined in the potential well.

Summary:- ΔE gap \propto Q.N. $\cdot \pi$, the reflecting the quantization of energy levels in quantum mechanics. This behavior underscores the quantum mechanical nature of the system. This behaves the wave-particle duality of quantum particles.

Que. 6 We need to apply the operator to the f^n and check if it equals a constant times the original function. The eigenvalue is the constant of probability

$$\frac{\partial}{\partial x} (\sin \theta \cdot \cos \theta) = \sin \theta \frac{\partial}{\partial \theta} \cos \theta + \cos \theta \frac{\partial}{\partial \theta} \sin \theta$$

1st derivative

$$= \sin \theta (-\sin \theta) + \cos \theta \cos \theta$$

$$\frac{\partial}{\partial x} \sin \theta \cos \theta = \cos^2 \theta - \sin^2 \theta \quad \text{--- (1)}$$

2nd derivative

$$\frac{\partial^2}{\partial x^2} (\sin \theta \cdot \cos \theta) = -2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) \quad \text{--- (2)}$$

Now let's apply the given operator to the f^4

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} + 6 \sin^2 \theta \right) \sin \theta \cos \theta$$

wrong

in question it is given $\left(\frac{\partial}{\partial x} \right)$

$$\begin{aligned} \Rightarrow & \sin \theta (-2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta)) + 6 \sin^2 \theta (\sin \theta \cos \theta) \\ = & -2 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta) \sin \theta \cos \theta + 6 \sin^2 \theta \sin \theta \cos \theta \\ = & -2 \sin^2 \theta \cos^2 \theta + 2 \sin^4 \theta + 6 \sin^3 \theta \cos \theta \end{aligned}$$

Now let's simplify the expression

$$\begin{aligned} & = -2 \sin^2 \theta \cos^2 \theta + 2 \sin^4 \theta + 6 \sin^3 \theta \cos \theta \\ & = -2 \cancel{\sin^2 \theta} \cos^2 \theta + 2 \cancel{\sin^2 \theta} \cdot \cos^2 \theta + 6 \sin^3 \theta \cos \theta \\ & = \boxed{6 \sin^3 \theta \cos \theta} \end{aligned}$$

So we can see there is no constant times the original f^4 .
 There $\sin \theta \cos \theta$ is not an eigenⁿ for given operator.

Que 7

$$|\psi\rangle = \alpha|10\rangle + \beta|11\rangle \quad |V\rangle, |VT\rangle \text{ basis}$$

$$|V\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

To find the probability to measuring the state $|V\rangle$

Born rule:- States that the probability of obtaining a specific outcome in a given by sq² of the absolute value of the inner product of the quantum state and the basis that we want to measure

$$P(|V\rangle) = |\langle V|\psi\rangle|^2$$

$$\langle V|\psi\rangle = \langle V|(\alpha|10\rangle + \beta|11\rangle) = \alpha\langle V|10\rangle + \beta\langle V|11\rangle$$

Given $|V\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

We need to calculate the inner products

$$\langle V|10\rangle = \frac{1}{\sqrt{2}}\langle 010\rangle + \frac{1}{\sqrt{2}}\langle 110\rangle = \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 = \frac{1}{\sqrt{2}}$$

$$\langle V|11\rangle = \frac{1}{\sqrt{2}}\langle 011\rangle + \frac{1}{\sqrt{2}}\langle 111\rangle = \frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

Substituting these values to $\langle V|\psi\rangle$

$$\langle V|\psi\rangle = \alpha\left(\frac{1}{\sqrt{2}}\right) + \beta\left(\frac{1}{\sqrt{2}}\right) = \frac{\alpha + \beta}{\sqrt{2}}$$

Now we can find the probability of measuring $|V\rangle$:

$$P(|V\rangle) = \left(\frac{\alpha + \beta}{\sqrt{2}}\right)^2 = \frac{(\alpha + \beta)^2}{2}$$

So the probability of measuring $|V\rangle$ is $\frac{(\alpha + \beta)^2}{2}$

Solⁿ Q) To find the probability of the particle having linear momentum $\pm \hbar k$, we need to calculate the probability density associated with this momentum.

The probability density is given by $|\psi(x)|^2$

$$|\psi(x)|^2 = (\cos \theta e^{ikx} + \sin \theta e^{-ikx})^2$$

$$|\psi(x)|^2 = (\cos \theta e^{ikx} + \sin \theta e^{-ikx})(\cos \theta e^{ikx} + \sin \theta e^{-ikx})$$

$$= \cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta$$

$$= 1 + 2 \sin \theta \cos \theta$$

To be 25% certain that particle has linear momentum $(\pm \hbar k)$

$$P(\pm \hbar k) = \int_{-\infty}^{\infty} (1 + 2 \sin \theta \cos \theta) dx$$

25%

$$\frac{1}{4} = 1 + 2 \sin \theta \cos \theta$$

$$-\frac{3}{4} = 2 \sin \theta \cos \theta$$

$$-\frac{3}{4} = \sin 2\theta$$

$$\theta = \frac{\sin^{-1}(-0.75)}{2}$$

$$\boxed{\theta \approx 60^\circ}$$

$$\therefore \sin(\pm \hbar k)$$

$$|\cos \theta|^2 = \frac{1}{4}$$

$$\cos \theta = \frac{1}{2}$$

$$\boxed{\theta = 60^\circ}$$

$$\theta = \frac{\pi}{3} = 0.5$$

$$\theta = \frac{5\pi}{3} = -0.5$$

Solⁿ-9 (i) The energy of n^{th} level is given by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

The e^- undergoes a transition from the $n=3$ level to $n=6$ and absorbing a photon of wavelength $\lambda = 500 \text{ nm}$

$$\Delta E = E_{n=6} - E_{n=3}$$

$$\Delta E = \frac{6^2 \pi^2 \hbar^2}{2mL^2} - \frac{3^2 \pi^2 \hbar^2}{2mL^2}$$

$$\Delta E = \frac{27 \pi^2 \hbar^2}{2mL^2}$$

We also know that the energy of a photon is related to its wavelength by

$$E_{\text{photon}} = \frac{hc}{\lambda}$$

$$E_{\text{photon}} = \Delta E$$

In case of photon absorbed, its energy matches the energy difference b/w two level

$$\frac{hc}{\lambda} = \frac{27 \pi^2 \hbar^2}{2mL^2}$$

$$L = \sqrt{\frac{27 \pi^2 \hbar^2}{2mc\lambda}}$$

$$L = \sqrt{\frac{27 \pi^2 \times (10^{-34})^2 \times 0.5}{2 \times 9.11 \times 10^{-31} \times 3 \times 10^8 \times 500 \times 10^{-9} \text{ m}}}$$

$$L \approx 2.42 \times 10^{-10} \text{ m} \approx 0.242 \text{ nm}$$